STUDENTS' UNDERSTANDING OF SQUARE NUMBERS AND SQUARE ROOTS

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Despite their apparent simplicity, the concepts of square numbers and square roots are problematic for high school students. I inquired into students' understanding of these concepts, focusing on obstacles that students face while attempting to solve square number problems. The study followed a modified analytic induction methodology that included a written questionnaire administered to 51 grade 11 students and follow up clinical interviews with 9 students. The study revealed significant obstacles relating to the representation of square numbers and confusion of concepts including both weak distinction between the concepts of square numbers and square roots and inconsistent evoking of their concept images.

INTRODUCTION AND FOCUS

Some mathematical concepts appear too simple to cause confusion for students. These concepts are taught as if to understand them only requires being informed of their pertinent properties and from then on no confusion should be possible. Square numbers are such a concept. What could be simpler than arranging dots into a square shape? And yet students do have a variety of ways of comprehending square numbers, and square numbers are not quite as simple as might appear at first glance.

In this study I investigate the research question: "What obstacles do students encounter when attempting to solve problems with square numbers and square roots? In particular to what degree does: confusion of concepts, and representation of square numbers and square roots, hamper students attempting to solve problems?"

Research that focuses on students' learning and understanding of 'simple' square numbers and square roots is slim. However, Gough (2007) does discuss the difficulties of teaching square roots and argues that in the case of square roots, the vocabulary can be confusing and detrimental to student understanding. 'Square number' and 'square root' are similar sounding phrases that evoke images from our everyday English language use of those words and while 'square number' *may* yield a useful image, 'square root' does not convey much meaning in and of itself. These two phrases are very similar and may hinder students when they are attempting to distinguish between the two. The role of definitions has not been studied with particular respect to square numbers or square roots, but the similarity of the terms square number and square root may be an obstacle for students.

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THEORETICAL FRAMEWORK

The data analysis was performed through the lens of the theoretical constructs of concept image and concept definition, and opaque and transparent representation. Each construct is described in general and with particular emphasis on square numbers and square roots.

Concept Image and Concept Definition

Tall and Vinner (1981) were the first to describe concept image and concept definition using these terms; they describe concept image as "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures" (p. 152). They also describe concept definition as "a form of words used to specify that concept. It may be learnt by an individual in a rote fashion or more meaningfully learnt and related to a greater or lesser degree to the concept as a whole" (p. 152). As an individual's concept image is built up of many parts and is developed through experience, some portions of the concept image may be incorrect or incomplete, and may conflict with that person's concept definition.

I looked for examples of students demonstrating a robust and multi-dimensional concept image and examples of students demonstrating a shallow concept image of square numbers and square roots. Examples of incomplete concept images of square numbers include statements that 'anything squared' is a square number or that 'perfect cubes cannot be perfect squares'. A more complete concept image would correctly limit the domain to integers as 'an integer times itself'.

Opaque and Transparent Representation

A number may be represented in numerous ways. Representations may be referred to as opaque or transparent; representations that highlight a desired property of a number may be said to be transparent to that property, while representations that obscure a property are said to be opaque to that property (Lesh, Behr & Post, 1999). Zazkis and Gadowsky (2001) assert that all representations are opaque to some features and transparent to others. With respect to square numbers and square roots, the representation of an expression may be more or less opaque to the feature of 'squareness'. 64^2 is an example of an expression that is transparent with respect to squareness. Here the exponent '2' clearly shows that this number is a square number based on the definition and common description of a square. The expression 8^4 is more opaque and less transparent than the first example, but the squareness is still somewhat evident if only attending to the exponent in the expression. 16^3 is now very opaque as the exponent shows no sign of the expression being a square number and the square number must be found by attending to the base of the expression to discover the square. Note that all of these are different representations of the number 4096, which is now very opaque to the squareness.

The role of representation dealing with numbers, rather than algebraic or geometric or other mathematical representations, has been well documented with respect to prime numbers (Zazkis & Liljedahl, 2004; Zazkis, 2005), irrational numbers, (Zazkis & Sirotic, 2004; Zazkis, 2005), divisibility and prime factorization (Zazkis & Campbell, 1996; Zazkis, 2008), but not with square numbers or square roots.

The lack of transparent representation has been found to be hindrance to students when solving problems or attempting to generate examples. However, if students do not have a clear understanding of the structure of a problem or expression, a transparent representation will not guarantee success. Zazkis and Sirotic (2004) found that only 60% of respondents gave the correct response to the question of whether 53/83 was rational or irrational after performing the division on a calculator. Although the representation of the number was transparent to rationality a large proportion of respondents did not attend to this representational feature. These students were not attending to the rational expression, but were focused instead on the partial decimal representation shown by their calculators. While studying divisibility, Zazkis and Campbell (1996) found that students must have sufficient understanding of the multiplicative structure in order to attend to the transparent features of an analogous problem.

METHODOLOGY

This study followed the methodology of modified analytic induction, as laid out by Bogdan and Biklen (1998). Modified analytic induction requires a phenomenon of interest and a working hypothesis or theory. One develops a loose descriptive theory, collects data and then recursively rewrites and modifies both the theory and even the phenomenon of interest to fit the new data. Modified analytic induction uses purposeful sampling in order to choose subjects that will facilitate the expansion of the developing theory.

In this study, the phenomenon of interest was students' understanding of square numbers and square roots, and in particular the obstacles that students encounter when attempting to solve problems with square numbers and square roots. The working theory that addresses these questions began as an assumption that the opaque representation of the expression would be an obstacle in students' ability to solve square number and square root problems.

The Participants

The participants in this study were 51 grade 11 students. The students ranged greatly in ability; the group included some of the most mathematically talented students in the school as well as students who were much less capable. Pre-calculus students were chosen for this study in order to capture students with both a great deal of familiarity with square numbers and a wide range of knowledge and ability.

The Instruments

Two instruments were used; a written questionnaire, completed by all participants and a follow-up semi-structured clinical interview designed to gain more insight into student responses from the questionnaire, completed by nine participants. The nine students who participated in the clinical interviews all had a grade of ‘A’ or ‘B’ in both their current mathematics course as well as their previous mathematics course. These students were selected through purposeful sampling by choosing students for the interview based on their willingness to explain their reasoning on the questionnaire responses and their willingness to attempt the problems in good faith.

The Tasks

The questionnaire tasks were designed to investigate students’ capacity to solve problems that moved from more transparent representations of square numbers through more opaque representations of square numbers. Two sample tasks are “Consider 36^2, 36^3, 36^4, 36^5, 36^6, 36^7. Circle the perfect squares.” and “How many perfect squares are there between 100 and 10,000?”

The clinical interviews were designed to gather additional information of a different nature than that gained from the questionnaire. While the questionnaire was designed to indicate *which* questions students had difficulty answering, the clinical interviews were designed to explore *why* students had difficulty answering a particular question.

RESULTS

The questionnaire revealed that students have a great deal of difficulty with opaque representations of square numbers. The sample task “Consider 36^2, 36^3, 36^4, 36^5, 36^6, 36^7. Circle the perfect squares.” was only answered correctly by two of 51 participants, while 19 students only circled 36^2, 17 circled 36^2, 36^4 and 36^6, and 3 participants circled only 36^2 and 36^4. Of the 51 participants, 39 or 76%, did not attend to the base of the expressions and were hampered by the opaque representation.

Concept Definition Confusion

Over the course of the study students used many terms, some interchangeably, and did not seem to have a strong sense of their definitions. During the clinical interviews I used the terms square number, perfect square and square root; the students used these terms as well as additional terms such as non-perfect square, perfect number and others. The meanings given by each student to the terms in use were often inconsistent, unclear or incorrect. These meanings were often only implicit or assumed, from the context, as participants rarely offered any definitions. The concept definitions that students had were not universal and did not seem to be well defined.

A prime example of a clear confusion between concept definitions can be found in my interview with Jack. Jack was an exceptionally able student who answered the majority of questions on the questionnaire and during the interview correctly and swiftly, but he often needed to ask for clarification if I wanted a perfect square or the square root if I used the term square number. At the end of the interview I asked explicitly for his definition of a square number, as I had noticed him using this term in an unusual manner previously in the interview.

Jack: Square number? Hmm. [pause] Umm, A whole number that is…that is being multiplied by itself to make a perfect square?

Jack was using the term square number to indicate the square root of a perfect square. He did not confuse perfect squares with square roots, but he was not sure about the term square number.

Another facet of the confusion with concept definitions came from Maya, who had been asked which of the following series are square numbers: $36^2, 36^3, 36^4, 36^5, 36^6, 36^7, 36^8$. She used her calculator and discovered that 36^3 is a square number and was very surprised.

Maya: Well because, what I understood of squares or perfect squares was that, well this would be a cube… wait that makes no sense cause it's a square still…but for a square what I thought they meant was like a 2D form…

Here Maya's definition of a square number was linked to her image of a square. In this case, her concept image of a square number was related to a geometric square, she was limited by this image in her mind and she had difficulty connecting it to the idea that a cube could also be a square number. Her concept image of a square number is narrower than it could be. Note also, the representation of 36^3 is transparent as to the number being a cube, but opaque to the number being a square. It is clear that Maya's definition of a square number did not rely on general factors or prime factors and may be quite different than that of her peers.

It is apparent from these examples that the definitions in use by the students are not always clear or consistent with mathematical conventions. Their definitions are also not locally consistent; these students do not share any definitions that are particular to their group. This may be due to the apparent lack of rigorous definitions supplied to students; students must therefore create their own concept definitions.

Inconsistent Concept Image

During the clinical interviews analysis, I found a larger conceptual problem than one of just unclear definitions; that is inconsistent evoking of concept image.

Kennedy was asked to find a perfect square larger than 500. After finding a square number larger than the target number, she became confused when the square root was smaller than the target number.

Kennedy: Um,…[pause] I guess the perfect square of 1000, would be 100,000? Like the square root of 100,000, maybe? Am I allowed to use a calculator?

[…]

Ok, so I'm doing the square root of 10,000. Which is 100, so wait, that's not bigger than 500.

This is not an instance of simply forgetting the original question and this type of confusion was not a unique event as shown by Rachel. She was asked for a perfect square that has three digits. Rachel gave 10,000 as her final answer, because 100 squared is 10,000. Throughout her interview Rachel repeatedly but inconsistently exchanged the term square number for the square root.

Another particularly clear case of this confusion came once again from Maya. Maya became confused between the square number and its square root during a task that asked her to find the number of square numbers between 0 and 100.

Maya:	Um, 2, no wait… the last one would be 10. Yeah, 2, 4, 9, …4?
Interviewer:	Ok so how did you come up with 4?
M:	Umm, well the last perfect square is 100, so the square root of a hundred is 10.
	Or, I don't know why I wrote that, but yeah its 10. Then you go back down to the next number, which is 9 that would be 81,…
	But then 8 doesn't have a square root, nor does 7, nor does 6, nor does 5, but 4 and 2 do have one. No wait, 2 does not have one. Or does it have one? No 2 does not have one. I'll go 3. [laughs]
	Or 1 wait. Is 1 a square root? I'm pretty sure 1 is a square root as well right?...I'm confused, hold on a second. Uh, yeah.

Maya was attempting to count the square numbers between 0 and 100; she gave her final answer on her paper as "4 – 1, 4, 9 and 100". She began counting at 10, the square root of 100. Her confusion began just after counting 9, the square root of 81, because "8 doesn't have a square root" even though she had been working on the *square* of 9 not the *square root* of 9.

Like Kennedy and Rachel, there is an issue of losing sight of the problem due to the labels Maya internally assigned to numbers. Maya had 10 and then 9 in her head as square roots, but the fact that 9 is also a square number seems to have confused her. When she moved down her list to 8, she should have squared it but she became confused because she knew that 8 is not a perfect square as 9 is. Maya evoked her own concept image of 9 as a square number when she should have been evoking the image of 9 as a square root. It is not so much Maya had a confused image of either square numbers or square roots, but that she became confused during the problem about what she was trying to accomplish. Maya and others demonstrated a difficulty coordinating concept images consistently with their work.

Representation

The opaque representation of square numbers was often an obstacle for students to overcome. The most common issue with representation was students who only attended to exponents in expressions when looking for square numbers. They did not attend to the base when determining if an expression was a square number, and in compound expressions that contained an exponent, any number without an exponent was treated as not a square number.

During the clinical interviews, each student was asked the following two questions: "Can k^3 ever be a square number?" and "Which of these numbers, $36^2, 36^3, 36^4, 36^5, 36^6, 36^7, 36^8$, are perfect squares?" However these questions were not always asked in this order. All participants who were asked about k^3 first claimed that it could not be a square number. However, students that were asked about the series first, before the question about k^3, usually manually checked all the expressions and like Maya were surprised to find that 36^3 was a perfect square. Subsequently, they were able to correctly answer that k^3.could be a perfect square for certain *k* when asked with that problem later in the interview.

When confronted with problems that forced students to attend to the base of a power expression, some students were more likely to attend to the bases in subsequent problems. However this was only common with problems that were very similar such as k^3 and 36^3. In unfamiliar problems most students continued to only attend to the exponents in the expressions, and were greatly hindered if the representation of a square number was not transparent in the exponent.

CONCLUSION

This study has found that students do experience difficulty when working with square numbers and square roots, and although the topics may seem simple, there is a wide variety of ways in which to think about and work with square numbers. Students face significant obstacles with opaque representation and concept confusion related to indistinct concept definitions and inconsistent evoking of concept images.

In particular, the students in this study did not share agreed-upon definitions for the terms involved with the wider mathematical community. Some students were unsure if square numbers must be squares of integers, or it they could be any number that could be 'square rooted', while others believed that 'square number' meant 'square root' as opposed to perfect square. I suggest that the lack of clear and concise definitions of square numbers and square roots given to students, is also an obstacle for students to overcome when attempting to solve square number problems.

A unique finding of this study is the confusion demonstrated between the concepts of square number and square root. Many other students became confused while working through a problem, as to which type of number they were dealing with and which properties those numbers had. There exists for them a difficulty in consistent evoking of the appropriate concept image. To what extent this confusion is prevalent and to what extent this confusion stems from the nature of the similarity of the terms square number and square root remains to be determined.

This study adds to the body of knowledge on the role of the representation of numbers, in this case square numbers and square roots. Opaque representation was found to be a large obstacle for students when attempting to solve problems with square numbers or square roots. When the representation was not transparent with respect to square numbers, students often claimed that the expression could not be a square number, without attempting to verify the statement. In this study students overwhelmingly did not attend to the base in exponential expressions.

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